Life of Fred

Linear Algebra

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Polka Dot Publishing

A Note to Classroom Students and Autodidacts

In calculus, there was essentially one new idea. It was the idea of the limit of a function. Using that idea, we defined the derivative and the definite integral. And then we played with that idea for two years of calculus.

In linear algebra, we dip back into high school algebra and begin with the idea of solving a system of linear equations like

$$\begin{cases} 3x + 4y = 18 \\ 2x + 5y = 19 \end{cases}$$

and for two or three hours on a Saturday while Fred goes on a picnic, we will play with that idea.

✓ We can change the variables: $\begin{cases} 3x_1 + 4x_2 = 18 \\ 2x_1 + 5x_2 = 19 \end{cases}$

✓ We can consider three equations and three unknowns.

✓ We can look at the coefficient matrix $\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$

✓ We can consider the case in which the system has exactly one solution (Chapter One), or when it has many solutions (Chapter Two), or when it has no solution (Chapter Three).

✓ Etc.

Of course, the "Etc." covers all the new stuff such as model functions, orthogonal complements, and vector spaces. But they are all just ideas that you might encounter in the Kansas sunshine as you went on a picnic.

Enjoy!

A Note to Teachers

Is there some law that math textbooks have to be dreadfully serious and dull? Is there a law that students must be marched through linear algebra shouting out the cadence count Definition, Theorem and Proof,

Definition, Theorem and Proof,

Definition, Theorem and Proof,

Definition, Theorem and Proof,

as if they were in the army. If there are such laws, then *Life of Fred: Linear Algebra* is highly illegal.

Besides being illegal, this book is also fattening. Instead of heading outside and going skateboarding, your students will be tempted to curl up with this textbook and read it. In 352 pages, they will read how Fred spent three hours on a Saturday picnic with a couple of his friends. I think that Mary Poppins was right: a spoonful of sugar can make life a little more pleasant.

So your students will be fat and illegally happy.

But what about you, the teacher? Think of it this way: *If your students are eagerly reading about linear algebra, your work is made easier. You can spend more time skateboarding!*

This book contains linear algebra—lots of it. All the standard topics are included. A good solid course stands admixed with the fun.

At the end of every chapter are six sets of problems giving the students plenty of practice. Some are easy, and some are like:

> If $T, T' \in \mathrm{Hom}(V, V)$, and if TT' is the identity homomorphism, then prove that $T'T$ is also the identity homomorphism.

Life of Fred: Linear Algebra also has a logical structure that will make sense to students. The best teaching builds on what the student already knows. In high school algebra they (supposedly) learned how to solve systems of linear equations by several different methods.

The four chapters that form the backbone of this book all deal with systems of linear equations:

Chapter One—Systems with Exactly One Solution
Chapter Two—Systems with Many Solutions
Chapter Three—Systems with No Solution
Chapter Four—Systems Evolving over Time

These chapters allow the students to get their mental meat hooks into the less theoretical material.

Then in the interlarded chapters (1½, 2½, 2¾, 3½) we build on that foundation as we ascend into the more abstract topics of vector spaces, inner product spaces, etc.

Lastly, your students will love you even before they meet you. They will shout for joy in the bookstore when they discover you have adopted a linear algebra textbook that costs only $49.

Contents

Chapter One

Systems of Equations with One Solution

$Ax = b$ ☺

Fred had never really been on a lot of picnics in his life. Today was special. Today at noon he was going to meet his two best friends, Betty and Alexander, on the Great Lawn on campus, and they were going to have a picnic.

One good thing about being at KITTENS University✶ is that just about everything imaginable is either on campus or nearby.

Wait! Stop! I, your loyal reader, need to interrupt. In your old age, dear author, you're getting kind of foggy-brained.

What do you mean?

I'm reading this stuff very carefully, since it's a math book and I have to pay attention to every word. Isn't it obvious that KITTENS would have "just about everything imaginable . . ." since you are doing the imagining?

Good point. I spoke the truth and plead as John Peter Zenger pleaded.✶✶

I accept. Please go on with your story.

Thank you.

Fred knew that food is one important part of a picnic. He picked up the local newspaper and read . . .

✶ KITTENS University. Kansas Institute for Teaching Technology, Engineering and Natural Sciences.

Background information: Professor Fred Gauss has taught math there for over five years. He is now six years old. Betty and Alexander are students of his. They are both 21.

✶✶ In his *Weekly Journal*, Zenger criticized the New York governor. Heavens! The government sent him to jail for libel. He had to wait ten months for his trial. At his trial in 1735 he was accused of promoting "an ill opinion of the government." Zenger's defense was that what he had written was true. The judge said that truth is no defense in a libel case. But the jury ignored the judge and set Zenger free. That marked a milestone in American law. Truth then became a legitimate defense in criminal libel suits in America after that trial. In England that idea did not catch on until the 1920s.

THE KITTEN Caboodle

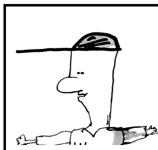
The Official Campus Newspaper of KITTENS University

Saturday 11:02 a.m. Grocery Shopping Edition 10¢

PICNIC MANIA— THE NEW RAGE

KANSAS: A new fad is sweeping the country. Everyone is going on picnics. This was announced last night on television.

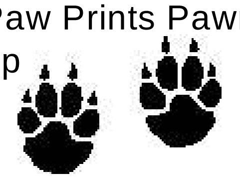
News of this great surge in popularity has taken the country by surprise. No one here at the Caboodle news center knew picnicking was popular, much less that it was the newest craze. (continued on p. 31)



"We must picnic," our university president declared in an exclusive Caboodle interview.

--- advertisement ---

Paw Prints Pawn Shop



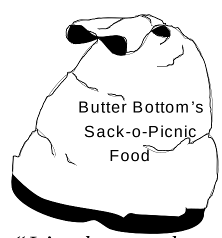
We are now accepting for pawn all the stuff from previous manias: Hula-Hoops™, things you stick in your ear to hear music, narrow neckties, leisure suits, laptops, ukeleles, raccoon coats, and overpriced real estate.

"Never fight the urge to pawn!" K

--- advertisement ---

Butter Bottom Foods

If you're new to picnicking, there's nothing that beats our . . .



"It's always a better buy at Butter Bottom!" K

Perfect! thought Fred. I'm sure that Butter Bottom's Sack-o-Picnic Food will do the trick. I don't want to disappoint Betty and Alexander.

In a jiffy,✶ Fred walked to Butter Bottom Foods. And there at the front of the store was a Sack-o-Picnic Food display.

✶ *In a jiffy* (or *in a jiff*) used to be a common expression meaning "in a very short period of time." Those fun-loving physicists have redefined a *jiffy* as the time it takes for light to travel the radius of an electron.

Wow! Fred thought. They sure make it easy. All I gotta do is choose a sack, and I'm ready to head off to see Alexander and Betty.

Fred was curious. He opened the first sack and looked inside. There were a can, five bottles, and three jars.

He took out the can . . . a bottle . . . and a jar.

In the first sack, one can, five bottles, and one jar cost $2.65.

$$c + 5b + 3j = 2.65$$

Fred knew, That's not enough to tell me what each of the items cost.

He opened the second sack. Two cans, 3 bottles, and 4 jars.

$$2c + 3b + 4j = 2.75$$

The third sack: Five cans, 32 bottles, and 3 jars. Wow. That's a lot of Sluice!

$$5c + 32b + 3j = 10.20$$

With three equations and three unknowns, Fred could use his high school algebra and solve this system of equations:

$$\begin{cases} c + 5b + 3j = 2.65 \\ 2c + 3b + 4j = 2.75 \\ 5c + 32b + 3j = 10.20 \end{cases}$$

In high school algebra, we were often more comfortable using x, y, and z:

$$\begin{cases} x + 5y + 3z = 2.65 \\ 2x + 3y + 4z = 2.75 \\ 5x + 32y + 3z = 10.20 \end{cases}$$

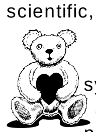
Intermission

Do your eyes begin to glaze over when you see lines of equations? In other words: Are you normal?

Unfortunately, much of linear algebra is about solving systems of linear equations like those above. This book has four main chapters. The first three chapters deal directly with solving systems of linear equations:

Chapter One: Systems with One Solution
Chapter Two: Systems with Many Solutions
Chapter Three: Systems with No Solution.

The crux of the matter is that systems of linear equations keep popping up all the time, especially in scientific, business, and engineering situations.



Who knows? Maybe even in your love life systems of linear equations might be waiting right around the corner as you figure the cost of 6 pizzas, 2 violinists, and 3 buckets of flowers.

On the next page is a Your Turn to Play. Even though this is the twelfth book in the *Life of Fred* series, the Your Turn to Play may be new to some readers. Let me explain what's coming.

I, as your reader, would appreciate that. I hate surprises.

Psychologists say that the best way to really learn something is to be personally involved in the process. The Your Turn to Play sections give you that opportunity.

The most important point is that you honestly attempt to answer each of the questions before you look at the solutions. Please, please, please, please, please, please, please, please, please, please, please with sugar on it.

Your Turn to Play

1. We might as well start off with the eyes-glaze-over stuff. Pull out your old high school algebra book if you need it. Solve

$$\begin{cases} c + 5b + 3j = 2.65 \\ 2c + 3b + 4j = 2.75 \\ 5c + 32b + 3j = 10.20 \end{cases}$$

by the "elimination method." (The other two methods that you may have learned are the substitution method—which works best with two equations and two unknowns—and the graphing method—which, in this case, would involve drawing three planes on the x-y-z axes and trying to determine the point of intersection.)

2. Since this is *linear* algebra, we will be solving *linear* equations. Which of these equations are linear?

$9x + 3y^2 = 2$

$3x + 2xy = 47$

$7\sin x + 3y = -8$

$5\sqrt{x} = 36$

3. Sometimes linear equations might have four variables. Then they might be written $3w + 2x + 898y - 5z = 7$.

But what about in the business world? In your fountain pen factory, there might be 26 different varieties of pens. Then your linear equation might look like: $2a + 6b + 8c + 9d - 3e - f + 2g + 14h + 4i - 3j + 2k - 11l + 3m + 8n + 20o + 5p + 2q + 3r + 9s + 2t + 99u + 3v + 8w + 8x - y + 2z = 98723$. Even then, we might get into a little trouble with the 11l (eleven "el") term or the 20o (twenty "oh") term.

If you are in real estate, and there are 40 variables involved in determining the price of a house (e.g., number of bedrooms, size of the lot, age of the house . . . .), you could stick in some of the Greek letters you learned in trig: $2a + 6b + 8c + 9d - 3e - f + 2g + \ldots + 8w + 8x - y + 2z + 66\alpha - 5\beta + 2\gamma + \ldots = \$384,280$.

If you are running an oil refinery, there might be a hundred equations. Then you might dip into the Hebrew alphabet (a b c) and the Cyrillic alphabet (Д, Ж, И).

One of the major thrusts of linear algebra is to make your life easier. Certainly, 6x + 5c – 9 Д + 2ξ = 3 doesn't look like the way to go.

Can you think of a way out of this mess?

4. How many solutions does $x + y = 15$ have?

5. [Primarily for English majors] What's wrong with the definition: "A linear equation is any equation of the form $a_1x_1 + a_2x_2 + \ldots a_nx_n = b$ where the a_i (for i = 1 to n) and the b are real numbers and n is a natural number"?

Recall, the natural numbers are {1, 2, 3, . . .} and they are often abbreviated by the symbol ℕ.

Complete Solutions

1. Now I hope that you hauled out a sheet of paper and attempted this problem before looking here. I know it's *easier* to just look at my answers than to do it for yourself.

And it's easier to eat that extra slice of pizza than to diet.
And it's easier to cheat on your lover than to remain faithful.
And it's easier to sit around than to do huffy-puffy exercise.
But *easier* can make you fat, divorced, and flabby.

My solution may be different than yours since there are several ways to attack the problem. However, our final answers should match.

I'm going to use x, y, and z instead of c, b, and j. The letter x will stand for the cost of one can of Picnic Rice™, y will stand for the cost of one bottle of Sluice™, and z will stand for the cost of a jar of mustard.

$$\begin{cases} x + 5y + 3z = 2.65 \\ 2x + 3y + 4z = 2.75 \\ 5x + 32y + 3z = 10.20 \end{cases}$$

If I take the first two equations, multiply the first one by –2, and add them together I get

$$-7y - 2z = -2.55$$

If I take the first and third equations, multiply the first one by –5, and add them together I get

$$7y - 12z = -3.05$$

Now I have two equations in two unknowns. If I add them together I get one equation in one unknown

$$-14z = -5.60$$

so $z = 0.40$ (which means that a jar of mustard costs 40¢).

The last part of the process is to **back-substitute**. Putting $z = 0.40$ into

$$7y - 12z = -3.05$$

we get $$7y - 12(0.40) = -3.05$$

so $y = 0.25$ (which means that a bottle of Sluice costs 25¢).

Back-substituting $z = 0.40$ and $y = 0.25$ into any one of the three original equations, will give $x = 0.20$ (so a can of Picnic Rice costs 20¢). This may be the last time you ever have to work with all those x's, y's, and z's (unless, of course, you become a high school math teacher). As we progress in linear algebra, the process of solving systems of linear equations will become easier and easier. Otherwise, why in the world would we be studying this stuff?

2.

$9x + 3y^2 = 2$ is not linear because of the y^2.

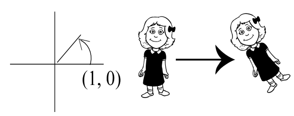
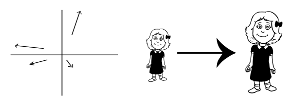
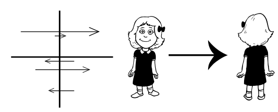
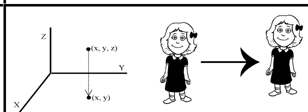
$3x + 2xy = 47$ is not linear because of the $2xy$.

$7\sin x + 3y = -8$ is not linear because of the $\sin x$.

$5\sqrt{x} = 36$ is not linear because of the $\sqrt{x}$.

3. The place where we dealt with an arbitrarily large number of variables was in *Life of Fred: Statistics*, but you might not remember the Wilcoxon Signed Ranks Test in which we had a sample $x_1, x_2, x_3, x_4. \ldots$ We used variables with subscripts. Now it doesn't make any difference whether we have three variables or 300.

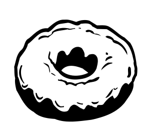
And you'll never have to face $6\text{c} - 8Э + 2\psi = 98.3$ unless you really want to.

Handy-Dandy Catalog of Linear Transformations		
Types	Picture	Example
Rotate each point of R^2 through an angle of θ		when $\theta = 45^{o}$ T((1, 0)) $= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$
Dilate the graph by doubling the distance from the origin		T((3, 4)) = (6, 8)
Reflect each point through an axis		T((2, 7)) = (–2, 7)
Project each point into a lower dimension		T((4, 5, 6)) = (4, 5)
Take the derivative		$T(4x^7) = 28x^6$
Multiply by a matrix	T(v) = Ax	

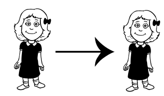
This chart is true, but it is misleading.
It's not misleading because I left out some linear transformations.

There are approximately 334,901,655 linear transformations that weren't mentioned, such as . . .

✓ The donut-hole transformation which takes a vector space of donuts and turns each vector into a doughnut hole. T(v) = 0 for all v. Traditional textbooks call this the **zero transformation**.



✓ The leave-it-alone transformation which maps each vector into itself. T(**v**) = **v** for every **v**. Traditional textbooks call this the **identity transformation**.



✓ The integration linear transformation that maps $\mathcal{P}_3$ to $\mathcal{P}_4$ defined by $T(\mathbf{v}) = \int_{x=0}^{x} \mathbf{v}\, dx$ where **v** is a polynomial in $\mathcal{P}_3$.✶

Okay. I, your reader, give up. What's so terrible about that Handy-Dandy Chart you have on the previous page? You said it was misleading.

It is misleading. Two pages ago I showed that every n×m matrix could give me a linear transformation. You give me **A**, and that will define T.

And you even called that a "major class of linear transformations."

I'm changing my mind.

Big deal. We'll call it a minor class of linear transformations.

No. No. No. We need to go in the other direction. Every linear transformation on finite dimensional vector spaces can be represented by a matrix multiplication. There is no other type of linear transformation. The chart should have looked like:

Handy-Dandy Catalog of All Linear Transformations		
Type	Picture	Example
Multiply by a matrix	T(**v**) = **Ax**	every T is an example

✶ The multiple roles of x in $\int_{x=0}^{x} \mathbf{v}\, dx$ may cause a little consternation. We have to integrate with respect to x (the "dx") because **v** is a polynomial in x. That x acts as a dummy variable. I would much rather have defined T(v) to equal $\int \mathbf{v}\, dx$. The only problem with that is that T wouldn't be a function! The indefinite integral $\int \mathbf{v}\, dx$ always has an answer with a "+ C" attached. $\int_{x=0}^{x} \mathbf{v}\, dx$ gets rid of the + C problem.

Index

Index

Index

Index

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and see what other books have been written about him

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